*Int. J. Solids Structures*, 1971, Vol. 7, pp. 1573 to 1579. Pergamon Press. Printed in Great Britain

# A NOTE ON THE STRAIN ENERGY OF ELASTIC SHELLS

FRITHIOF I. NIORDSON

Department of Solid Mechanics, The Technical University of Denmark, Lyngby, Denmark

**Abstract**—The errors inherent in Love's uncoupled strain energy expression are estimated on the basis of general considerations regarding physical dimensions and properties of invariance, especially with respect to inversion of the normal to the middle surface.

## INTRODUCTION

THERE seem to be two fundamentally different approaches to shell theory, both of which originated historically in the special case of elastic plates. The first was due to Poisson and Cauchy and is essentially the three-dimensional theory of elasticity adapted to the geometrical shape of shells. This theory can—at least in principle—yield exact solutions in the sense of three-dimensional elasticity. The second, and the one that is of interest to us here, was originally due to Kirchhoff.† It has proved to be a fruitful theory and is widely used in mechanics and engineering today. The basic concept on which Kirchhoff's theory is founded seems to be contained in the following assumption:

*The kinematics of the middle surface of a shell determine the state of the three-dimensional body as a whole with sufficient accuracy.*

Any theory based axiomatically on this assumption is really a two-dimensional theory analogous to the one-dimensional beam theory first conceived by Euler.

The object of this paper is to analyse the strain energy expression for a shell made of an isotropic and homogeneous material, obeying Hooke's law under the basic assumption stated above. Using dimensional analysis and properties of invariance, it is found that Love's strain energy expression has inherent errors of, at most, the relative order $h/R$ or $(h/L)^2$, depending on which is critical.‡ This estimate agrees with the results of Novozhilov [2], Novozhilov and Finkel'shtein [3] and Koiter [4] but has wider implications since no *further* assumptions (like the Kirchhoff assumption regarding points on a normal to the middle surface or plane stress) are involved. The importance of normal inversion for investigating invariance of the strain energy expression was apparently first observed by Serbin [5], who *assumes*, however, that the strain energy expression is independent of the initial curvature of the shell.

No estimate of errors introduced through the basic assumption itself is attempted. In fact, no such estimate is generally possible without restricting the class of permissible load distributions. This follows from the fact that there exist essentially different solutions to the exact three-dimensional differential equations of elasticity exhibiting identical deformation patterns in the middle surface. On the other hand, there is no intention here to restrict the

† For an account of the development of the theory of elastic plates and shells, see Todhunter and Pearson [1].

‡ Here $h$ is the thickness of the shell, $R$ the numerical value of the smallest principal radius of curvature and $L$ a characteristic wave length of the deformation pattern of the middle surface.

class of permissible load distributions or, which amounts to the same thing, to relate the error to parameters characterizing the load distribution. The immediate physical significance of the basic assumption that guides us in selecting cases in which a shell theory based on it may be applied with success, may justify an effort to estimate the magnitude of errors introduced through further assumptions.

## THE STRAIN ENERGY DENSITY

The deformation of the middle surface is uniquely determined by the tangent and normal displacements $v^\alpha$ and $w$ as functions of the coordinates $u^\alpha$ of this surface. The intrinsic deformation is given by the increments of the first and second fundamental tensors $a_{\alpha\beta}$ and $d_{\alpha\beta}$, viz.

$$E_{\alpha\beta} = \tfrac{1}{2}(v_{\alpha,\beta}+v_{\beta,\alpha})-d_{\alpha\beta}w$$

$$K_{\alpha\beta} = w_{,\alpha\beta}+d_{\alpha\gamma}v^{\gamma}_{,\beta}+d_{\beta\gamma}v^{\gamma}_{,\alpha}+v^{\gamma}d_{\gamma\alpha,\beta}-d_{\beta\gamma}d^{\gamma}_{\alpha}w$$

where $E_{\alpha\beta}$ is the strain tensor and $K_{\alpha\beta}$ the bending tensor. Comma denotes covariant derivation.

The second fundamental tensor $d_{\alpha\beta}$ and the normal displacement $w$ are defined in relation to the arbitrarily chosen positive direction of the normal vector to the middle surface. Thus, if the normal direction is reversed, $d_{\alpha\beta}$ and hence also $K_{\alpha\beta}$ change signs (but $a_{\alpha\beta}$ and $E_{\alpha\beta}$ do not). This fact will be explored below. The significance is, of course, that the way in which true invariants (like the strain energy) may depend on $d_{\alpha\beta}$ and $K_{\alpha\beta}$ is somehow restricted.

For a material obeying Hooke's law, the strain energy is necessarily a quadratic function of the measures of internal deformation.

The expression for the strain energy density $W$ can be written in the following dimensionless form

$$W/Eh = C^{\alpha\beta\gamma\delta}E_{\alpha\beta}E_{\gamma\delta}+D^{\alpha\beta\gamma\delta}E_{\alpha\beta}K_{\gamma\delta}+F^{\alpha\beta\gamma\delta}K_{\alpha\beta}K_{\gamma\delta}+r \tag{1}$$

where $E$ is Young's modulus, $h$ the (constant) thickness of the shell and the fourth order contravariant tensors $C^{\alpha\beta\gamma\delta}$, $D^{\alpha\beta\gamma\delta}$ and $F^{\alpha\beta\gamma\delta}$ are functions of the geometry of the shell ($a_{\alpha\beta}$, $d_{\alpha\beta}$ and $h$) and of Poisson's ratio $\nu$. The last term $r$ is a quadratic function of the (covariant) derivatives of $E_{\alpha\beta}$ and $K_{\alpha\beta}$ of all orders.

The purpose of this paper is to examine the strain energy expression (1) and to evaluate the errors introduced in simplifying it. Special interest is attached to the errors involved in using Love's strain energy expression

$$W/Eh = \frac{1}{2(1-\nu^2)}\{(1-\nu)E_{\alpha\beta}E^{\alpha\beta}+\nu E^{\alpha}_{\alpha}E^{\beta}_{\beta}\}+\frac{h^2}{24(1-\nu^2)}\{(1-\nu)K_{\alpha\beta}K^{\alpha\beta}+\nu K^{\alpha}_{\alpha}K^{\beta}_{\beta}\}$$

instead of the complete quadratic form (1).

Our examination will be based upon the following:

1. The strain energy is a positive definite function.
2. The strain energy is invariant with respect to inversion of the normal to the middle surface.
3. The strain energy is an analytical function of the invariants $(H, K, h, \nu)$, regular at $h = 0$, $H = 0$ and $K = 0$, where $H$ and $K$ are the mean and Gaussian curvatures of the middle surface, respectively.

## ANALYSIS OF INVARIANCE AND PHYSICAL DIMENSIONS

The invariance of (1) with respect to coordinate transformations must be valid for any given state of deformation. However, this can only be so if *each term* on the right hand side of (1) is invariant. We may thus proceed to examine the terms one by one.

The properties of symmetry of $E_{\alpha\beta}E_{\gamma\delta}$ imply that only the corresponding symmetrical part of $C^{\alpha\beta\gamma\delta}$ contributes to the inner product. Taking advantage of this fact we can—without loss of generality—express the fourth order tensor $C^{\alpha\beta\gamma\delta}$ in the following form

$$C^{\alpha\beta\gamma\delta} = C_1 a^{\alpha\beta}a^{\gamma\delta}+C_2 a^{\alpha\gamma}a^{\beta\delta}+C_3 h a^{\alpha\beta}d^{\gamma\delta}+C_4 h a^{\alpha\gamma}d^{\beta\delta}+C_5 h^2 d^{\alpha\beta}d^{\gamma\delta}+C_6 h^2 d^{\alpha\gamma}d^{\beta\delta} \qquad (2)$$

where the coefficients $C_1, \ldots, C_6$ are *dimensionless* analytic functions of the *dimensionless* variables $\nu$, $Hh$ and $Kh^2$. There is apparently another possibility of getting dimensionless coefficients, which is essentially different, namely by using the reciprocal values of the invariants of curvature $1/H$ and $1/K$. However, as the product is regular at $H = K = 0$ we can only achieve regularity for the coefficients and their multiplying factors by using $h$ and $h^2$ as indicated. This is seen from the fact that $(a\cdot\cdot d\cdot\cdot E..E..)/H$ and $(d\cdot\cdot d\cdot\cdot E..E..)/H^2$ are not regular at $H = 0$ when $K \neq 0$ and similarly, that $(d\cdot\cdot d\cdot\cdot E..E..)/K$ is not regular at $K = 0$ when $H \neq 0$. Note that *inner* products of $d_{\alpha\beta}$ for instance $d^\alpha_\xi d^\xi_\eta d^{\eta\beta}$, can be reduced to first order tensors using the Hamilton–Cayley theorem for the second order tensor $d_{\alpha\beta}$:

$$d^\alpha_\beta d^\beta_\gamma = 2Hd^\alpha_\gamma - Ka^\alpha_\gamma$$

so that no inner products appear explicitly in (2).

When the direction of the normal vector is reversed, $d^{\alpha\beta}$ changes sign and $C_3$ and $C_4$ must therefore also change signs, while their absolute values remain unchanged. The coefficients $C_1, C_2, C_5$ and $C_6$ do not undergo any change. Clearly, this restricts the dependence of the $C$s on the variable $Hh$ and implies that

$$C_1(\nu, Hh, Kh^2) = \bar{C}_1(\nu, (Hh)^2, Kh^2)$$

and similarly for $C_2$, $C_5$ and $C_6$, whereas

$$C_3(\nu, Hh, Kh^2) = Hh\bar{C}_3(\nu, (Hh)^2, Kh^2)$$

and similarly for $C_4$.

It is now seen that the first term $C^{\alpha\beta\gamma\delta}E_{\alpha\beta}E_{\gamma\delta}$ of the strain energy expression may be considerably simplified if terms of relative order $(h/R)^2$ are omitted. As the functions $\bar{C}_1, \ldots, \bar{C}_6$ are regular at $(Hh)^2 = Kh^2 = 0$ we are justified in evaluating them at $(Hh)^2 = Kh^2 = 0$ within this approximation. In the following we shall denote the coefficients $\bar{C}_1, \ldots, \bar{C}_6$, evaluated at $(Hh)^2 = Kh^2 = 0$ by $C_{10}, \ldots, C_{60}$.

It will next be shown that the contribution from all terms except the first two terms of (2) are, at most, of the relative order $(h/R)^2$.

Let us multiply (2) through by $E_{\alpha\beta}E_{\gamma\delta}$. The first two terms yield

$$C_{10}(e_1+e_2)^2+C_{20}(e_1^2+e_2^2) \qquad (3)$$

where $e_1$ and $e_2$ are the principal strains ($|e_1| \geq |e_2|$), and where $C_{10}$ and $C_{20}$ have positive values due to the fact that the strain energy expression is positive definite even when $H = K = 0$ [i.e. when (3) is the complete form].

The contribution from the remaining terms, one by one, will be compared with the sum (3). As

$$|d^{\gamma\delta}E_{\gamma\delta}| \leq 2|e_1|/R$$

we have

$$|C_3ha^{\alpha\beta}d^{\gamma\delta}E_{\alpha\beta}E_{\gamma\delta}| \leq |C_{30}Hh^2 2e_1^2/R|$$

and, furthermore, as

$$|a^{\alpha\gamma}d^{\beta\delta}E_{\alpha\beta}E_{\gamma\delta}| \leq 4e_1^2/R$$

and

$$|d^{\alpha\gamma}d^{\beta\delta}E_{\alpha\beta}E_{\gamma\delta}| \leq 4e_1^2/R^2$$

we get the estimates

$$|C_4ha^{\alpha\gamma}d^{\beta\delta}E_{\alpha\beta}E_{\gamma\delta}| \leq |C_{40}Hh^2 4e_1^2/R|$$

and

$$|C_5h^2d^{\alpha\beta}d^{\gamma\delta}E_{\alpha\beta}E_{\gamma\delta}| \leq |C_{50}h^2(2e_1/R)^2|.$$

This shows that the relative error in omitting all terms except (3) is, at most, of the order $(h/R)^2$.

So far, we have made no provision for including higher order tensors than the second order fundamental tensors $a_{\alpha\beta}$ and $d_{\alpha\beta}$ on the right hand side of equation (2). Obviously, such tensors (representing the "waviness" of the undeformed middle surface) may in general be obtained by covariant derivation of $d^{\alpha\beta}$. As the tensor $C^{\alpha\beta\gamma\delta}$ is of even order, the order of covariant derivatives in any one term must add up to an even number. Hence, the lowest order terms contain either a second derivative $(d_{..,..})$ or the product of two first derivatives $(d_{..,.}d_{..,.})$. In either case, a factor $h^4$ appears when the coefficient is dimensionless. It will be seen that the relative order of magnitude of these lowest order terms is $(h/R)^2(h/L)^2$, where $L$ is a characteristic wave length of the wave pattern for the undeformed middle surface. However, as this wave length is of the order $R$, we have the estimate $(h/R)^4$ for the contribution of terms omitted in (2).

This concludes our analysis of the term $C^{\alpha\beta\gamma\delta}E_{\alpha\beta}E_{\gamma\delta}$ of the strain energy expression (1).

Let us now turn to the third term $F^{\alpha\beta\gamma\delta}K_{\alpha\beta}K_{\gamma\delta}$. As the sign of $K_{\alpha\beta}K_{\gamma\delta}$ is invariant with respect to normal inversion, this term can be treated in almost exactly the same manner as the first one. It is then easily checked that the contribution to the strain energy from the third term is given by the expression

$$h^2\{F_{10}(k_1+k_2)^2+F_{20}(k_1^2+k_2^2)\} \tag{4}$$

the terms omitted being at most of the relative order of $(h/R)^2$. Here $k_1$ and $k_2$ are the principal changes of curvature ($|k_1| \geq |k_2|$). The expression (4) is seen to be quite analogous to (3), however the factor $h^2$ appears here, compensating the physical dimensions of $k_1$ and $k_2$. Like $C_{10}$ and $C_{20}$, the coefficients $F_{10}$ and $F_{20}$ are dimensionless functions of $\nu$.

The mixed term $D^{\alpha\beta\gamma\delta}E_{\alpha\beta}K_{\gamma\delta}$ has to be examined in more detail, since it is essentially different from the first term $C^{\alpha\beta\gamma\delta}E_{\alpha\beta}E_{\gamma\delta}$ and the third term $F^{\alpha\beta\gamma\delta}K_{\alpha\beta}K_{\gamma\delta}$ in that the sign of the mixed factor $E_{\alpha\beta}K_{\gamma\delta}$, and hence also the sign of $D^{\alpha\beta\gamma\delta}$, changes with the direction of the normal vector to the middle surface.

It will be seen that the tensor $D^{\alpha\beta\gamma\delta}$ can be written in the following form

$$D^{\alpha\beta\gamma\delta} = D_{10}Hh^2a^{\alpha\beta}a^{\gamma\delta}+D_{20}Hh^2a^{\alpha\gamma}a^{\beta\delta}+D_{30}h^2a^{\alpha\beta}d^{\gamma\delta}+D_{40}h^2a^{\gamma\delta}d^{\alpha\beta}+D_{50}h^2a^{\alpha\gamma}d^{\beta\delta}$$

where the terms omitted represent contributions to the strain energy of at most the relative order $(h/R)^2$. Multiplying through by $E_{\alpha\beta}K_{\gamma\delta}$ we get the sum

$$D_{10}Hh^2(e_1+e_2)(k_1+k_2)+D_{20}Hh^2E^{\alpha}_{\beta}K^{\beta}_{\alpha}+D_{30}h^2(e_1+e_2)d^{\alpha}_{\beta}K^{\beta}_{\alpha}$$
$$+D_{40}h^2(k_1+k_2)d^{\alpha}_{\beta}E^{\beta}_{\alpha}+D_{50}h^2d^{\alpha}_{\beta}E^{\beta}_{\gamma}K^{\gamma}_{\alpha}$$

and we proceed to compare the terms of this sum one by one with the sum of (3) and (4). For this purpose we shall use the elementary inequality

$$|2bxy| \le \frac{|b|}{\sqrt{(ac)}}(ax^2+cy^2) \qquad a>0, \quad c>0$$

which is valid for all $b$, $x$ and $y$.

For the first term we get the estimate

$$|D_{10}Hh^2(e_1+e_2)(k_1+k_2)| \le \frac{|D_{10}|}{2\sqrt{(C_{10}F_{10})}}\{C_{10}(e_1+e_2)^2+h^2F_{10}(k_1+k_2)^2\}|Hh|$$

which shows that it is, at most, of the relative order $|Hh|$.

For the second term $D_{20}Hh^2E^{\alpha}_{\beta}K^{\beta}_{\alpha}$ we have the inequality

$$|E^{\alpha}_{\beta}K^{\beta}_{\alpha}| \le 2|e_1k_1|$$

and thus

$$|D_{20}Hh^2E^{\alpha}_{\beta}K^{\beta}_{\alpha}| \le \frac{|D_{20}|}{2\sqrt{(C_{20}F_{20})}}\{C_{20}e_1^2+h^2F_{20}k_1^2\}|Hh|.$$

Similarly, as

$$|d^{\alpha}_{\beta}K^{\beta}_{\alpha}| \le 2|k_1|/R, \qquad |d^{\alpha}_{\beta}E^{\beta}_{\alpha}| \le 2|e_1|/R$$

and

$$|d^{\alpha}_{\beta}E^{\beta}_{\gamma}K^{\gamma}_{\alpha}| \le 4|e_1k_1|/R$$

we get the estimates

$$|D_{30}h^2(e_1+e_2)d^{\alpha}_{\beta}K^{\beta}_{\alpha}| \le \frac{|D_{30}|}{\sqrt{(C_{10}F_{20})}}\{C_{10}(e_1+e_2)^2+h^2F_{20}k_1^2\}(h/R)$$

$$|D_{40}h^2(k_1+k_2)d^{\alpha}_{\beta}E^{\beta}_{\alpha}| \le \frac{|D_{40}|}{\sqrt{(C_{20}F_{10})}}\{C_{20}e_1^2+h^2F_{10}(k_1+k_2)^2\}(h/R)$$

and

$$|D_{50}h^2d^{\alpha}_{\beta}E^{\beta}_{\gamma}K^{\gamma}_{\alpha}| \le \frac{2|D_{50}|}{\sqrt{(C_{20}F_{20})}}\{C_{20}e_1^2+h^2F_{20}k_1^2\}(h/R).$$

We may now conclude that the term $D^{\alpha\beta\gamma\delta}E_{\alpha\beta}K_{\gamma\delta}$ is, at most, of the relative order $h/R$.

Finally, let us examine the last term $r$ on the right hand side of the strain energy expression (1). This term represents a quadratic function of the covariant derivatives of $E_{\alpha\beta}$ and $K_{\alpha\beta}$ of all orders. There are a great number of possibilities for constructing quadratic invariants from the fundamental tensors $a^{\alpha\beta}$, $d^{\alpha\beta}$ and the covariant derivatives of $E_{\alpha\beta}$ and $K_{\alpha\beta}$ of given orders. However, it is easily seen that the two lowest order quadratic terms of the covariant derivatives of $E_{\alpha\beta}$ contain only products of the types $(E\ldots, E\ldots)$ and $(E\ldots E\ldots)$, since the resulting tensor has to be of even order.

Corresponding sixth order tensor coefficients multiplying these quadratic functions have the physical dimension (length)$^2$ and will thus contain the factor $h^2$. From this it follows that the relative order of magnitude of these terms is on an average $(h/L)^2$ in comparison with the term $C^{\alpha\beta\gamma\delta}E_{\alpha\beta}E_{\gamma\delta}$, where $L$ is a characteristic wave length of the deformation pattern on the middle surface.† Terms containing higher order derivatives of $E_{\alpha\beta}$ are of order $(h/L)^4$ or less.

The remaining terms of $r$ contain derivatives of $K_{\alpha\beta}$ and are compared with either $D^{\alpha\beta\gamma\delta}E_{\alpha\beta}K_{\gamma\delta}$ or $F^{\alpha\beta\gamma\delta}K_{\alpha\beta}K_{\gamma\delta}$, depending on whether the strain tensor $E_{\alpha\beta}$ (or its derivatives) is involved. It is immediately clear that, on an average $r$ is, at most, of the relative order $(h/L)^2$. Thus, we find that

$$W/Eh = C_{10}(e_1+e_2)^2+C_{20}(e_1^2+e_2^2)+h^2\{F_{10}(k_1+k_2)^2+F_{20}(k_1^2+k_2^2)\}. \tag{5}$$

## CONCLUSION

Under the basic assumption that the kinematics of the middle surface determine the state of the shell with sufficient accuracy, we have found that the uncoupled strain energy expression (5) has inherent relative errors of, at most, the orders $h/R$ and $(h/L)^2$.

The strain energy expression is a link between the two main sections (dealing with the kinematics and the statics, respectively) of a consistent two-dimensional theory of elastic shells, which are otherwise unconnected. For such a two-dimensional study, the result obtained has important implications. We can now derive a complete and consistent two-dimensional theory of elastic shells without introducing further assumptions and still retain *uncoupled* relations between the strain–bending measures and the force–couple measures.

The strain energy expression (5) contains four dimensionless coefficients $C_{10}$, $C_{20}$, $F_{10}$ and $F_{20}$, which cannot be determined from further considerations restricted to *two* dimensions. These coefficients may be determined by studying four selected special cases for which exact solutions of the three-dimensional equations of elasticity are available. The general form of the expression (5) may be used for treating sandwich shells, but assuming homogeneity in the normal direction, the coefficients of Love's strain energy expression are readily obtained.

## REFERENCES

[1] I. TODHUNTER and K. PEARSON, *A History of the Theory of Elasticity and of the Strength of Materials*, Vol. I, pp. 241–276, 336–357. Vol. II, part II, pp. 83–91. Cambridge University Press (1893).

[2] V. V. NOVOZHILOV, Certain remarks regarding the theory of shells. *Prikl. Mat. Mekh.* **3** (1941).

† Note, that the characteristic wave length $L$ can only be determined *a posteriori*.

[3] V. V. NOVOZHILOV and R. M. FINKEL'SHTEIN, Regarding the errors connected with the Kirchhoff hypotheses in the theory of shells. *Prikl. Mat. Mekh.* **VII** (1943).
[4] W. T. KOITER, A consistent First Approximation in the Theory of Thin Elastic Shells, *Proc. of the IUTAM Symp. on the Theory of Thin Elastic Shells*, Amsterdam, pp. 12–33 (1960).
[5] H. SERBIN, Quadratic invariants of surface deformations and the strain energy of thin elastic shells. *J. Math. Phys.* **4**, pp. 838–851 (1963).

(*Received* 30 *November* 1970; *revised* 3 *March* 1971)

**Абстракт**—Оцениваются неточности, встречающиеся в выражении несопряженной энергии деформации Лява, на основе общих сообраюсемий иоробця, относительно физических размеров и сьойств инвариантности, принимая во внимание инверсию нормальной к срединной поверхности.